

# Economic growth in the European Union modelled with fractional derivatives: first results

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**Abstract.** This paper presents models of economic growth for all states of the European Union (EU), since either 1970 or the year of accession to the EU. Both integer and fractional order models are obtained, where the gross domestic product (GDP) is a function of the country's land area, gross capital formation (GCF), exports of goods and services, and average years of school attendance.

**Key words:** economic growth, Europe, fractional calculus.

## 1. Introduction

Economic growth is conditioned by many factors, which act over time. This effect can be modelled using fractional derivatives, more accurately than using integer derivatives only [1–5]. In fact, it is reasonable to speak of the diffusion of several of the factors that condition economic growth, and of the diffusion of economic growth itself [6–8]. Diffusion processes in biological systems can often be modelled using fractional derivatives [9], and published results show that this also happens with financial models [10–23]. Fractional derivatives themselves have an economic interpretation [24] and are needed in the formulation of models for economic processes with long memory [25]. Fractional order models have been built for the GDP of several countries at a world level, both for recent years only and for longer time series [26, 27].

In previous papers we developed integer order and fractional order models, with the latter outperforming the former, for the economic growth of four economies in Western Europe, all bordering the Mediterranean [28, 29]: Portugal, Spain, France and Italy. In this paper we develop similar models for all states of the European Union (EU). This choice is motivated by the high degree of integration of the national economies involved, allowing to assume that similar patterns can be found in the resulting models. Continuous series of data are available from 1970 on; models are thus obtained in the 1970–2016 period for the states which in 1970 were members of the European Economic Community (EEC), predecessor of the EU, established in 1993 by the Maastricht Treaty. Models for other states are presented from the year of accession to the EEC or the EU (see Table 1).

The paper presents the methodology followed in Sec. 2 and the results obtained in Sec. 3. A discussion and conclusions are given in Sec. 4. The data employed for the models are tabulated in an Appendix.

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## 2. Methodology

The models considered have the following form for each of the member states of the EU:

$$y(t) = f(x_1, x_4, x_5, x_6). \quad (1)$$

The output model  $y$  is the GDP (in 2016 euros). The  $x_k$  are the variables on which the output depends:

- $x_1$ : land area (km<sup>2</sup>);
- $x_4$ : school attendance (years);
- $x_5$ : gross capital formation (GCF) (in 2016 euros);
- $x_6$ : exports of goods and services (in 2016 euros).

The rationale behind this choice of variables is the following:

- natural resources are represented by  $x_1$ ;
- the quality of human resources is represented by  $x_4$ ;
- the resources manufactured and the impact of investment on the economy are represented by  $x_5$ ;
- external impacts on the economy are represented by  $x_6$ .

The numeration of these variables is not consecutive, because they are a subset of those used in [28, 29], mentioned above in Sec. 1. The variables retained in this paper are those that were shown to be relevant for all the four models developed in those references.

The integer order model considered is

$$y(t) = C_1 x_1(t) + C_4 x_4(t) + C_5 \int_{t_0}^t x_5(t) dt + C_6 x_6(t), \quad (2)$$

where  $C_k$  are constant weights for each of the variables, and  $t_0$  is the first year considered. Notice that the accumulated

gross capital formation  $\int_{t_0}^t x_5(t) dt$  is used as a measure of manufactured resources.

Its generalization to non-integer orders is as follows:

$$y(t) = \sum_{k=1,4,5,6} C_k D^{\alpha_k} x_k(t), \quad (3)$$

where  $\alpha_k$  are the differentiation orders of each variable. The Caputo definition of fractional derivative  $D^{\alpha_k}$  was used [30].

Table 1  
Year of accession to the EEC (1957–1993) or the EU (1993–present) of all member states

Year	Accession of states
1957	Belgium (BEL), France (FRA), Germany* (DEU), Italy (ITA), Luxembourg (LUX), Netherlands (NDL)
1973	Denmark (DNK), Ireland (IRL), United Kingdom (GBR)
1981	Greece (GRC)
1986	Portugal (PRT), Spain (ESP)
1995	Austria (AUT), Finland (FIN), Sweden (SWE)
2004	Czech Republic (CZE), Cyprus (CYP), Estonia (EST), Hungary (HUN), Latvia (LVA), Lithuania (LTU), Malta (MLT), Poland (POL), Slovakia (SVK), Slovenia (SVN)
2007	Bulgaria (BGR), Romania (ROU)
2013	Croatia (HRV)

\* In 1990, the former German Democratic Republic was integrated into the Federal Republic of Germany. There was no increase in the number of member-states, but the EEC territory got larger, and variables for Germany have large variations in that year.

### 3. Results

This section contains the models for the economies of all states of the EU in the period between 1970 to 2016 (see economic data in Tables 4 and 5 in the Appendix).

The fitting procedure is implemented in MATLAB. Nelder-Mead's simplex search method (implemented in function *fminsearch*) is used to minimize the mean square error (MSE), given by

$$\text{MSE} = \frac{\sum_{j=1}^N (y_j - \hat{y}_j)^2}{N}. \quad (4)$$

Here  $N$  is the number of points, and  $y_j$  and  $\hat{y}_j$  are the real output and the model output, respectively. The MSE alone is not relied upon to evaluate the quality of the fit obtained by the resulting models: other performance indices were calculated as well. These were:

1. The mean absolute deviation (MAD), given by

$$\text{MAD} = \frac{\sum_{j=1}^N |y_j - \hat{y}_j|}{N}. \quad (5)$$

2. The coefficient of determination ( $R^2 \in (0, 1)$ ), given by

$$R^2 = 1 - \frac{\sum_{j=1}^N (y_j - \hat{y}_j)^2}{\sum_{j=1}^N (y_j - \bar{y})^2}. \quad (6)$$

Here  $\bar{y}$  is the mean of the GDP.

3. The  $t$ -values and  $p$ -values for each variable.

These are calculated with MATLAB command *regstats*.

As will be seen below, not all four variables  $x_1$ ,  $x_4$ ,  $x_5$  and  $x_6$  turned out to be necessary for every single model. This could be evaluated from the  $t$ - and  $p$ -values for each variable, by checking whether or not the performance indexes MAD and  $R^2$  deteriorate significantly when removing one or more variables from the model, and also using the Akaike Information Criterion (AIC):

$$\text{AIC} = N \log \frac{\sum_{j=1}^N (y_j - \hat{y}_j)^2}{N} + 2K + \frac{2K(K+1)}{N-K-1}. \quad (7)$$

Here  $K$  is the number of parameters of the model. The value of the AIC does not give information about the quality of a model. However, comparing the AIC values of different models, it can be seen which ones are more likely to be a good model for the data, as a lower value indicates a more likely model. Furthermore, if there are  $M$  models, the Akaike weight, given by

$$w_i = \frac{\exp \left( -\frac{\text{AIC}_i - \min_M \text{AIC}}{2} \right)}{\sum_{j=1}^M \exp \left( -\frac{\text{AIC}_j - \min_M \text{AIC}}{2} \right)}, \quad (8)$$

provides the probability of model  $i$  being the best of all the  $M$  models.

The results of the models for the several EU member-states are shown in Fig. 1 and 2, with performance indices in Table 2. In that table,  $t$ -values that correspond to variables necessary for the model, assuming a 5% significance level, are given in bold. The values of the orders  $\alpha$  and the coefficients  $C$  are given in Table 3 (notice that such orders for integer model (2) are  $\alpha_1 = 0$ ,  $\alpha_4 = 0$ ,  $\alpha_5 = -1$  and  $\alpha_6 = 0$ ).

### 4. Discussion and conclusions

As can be seen, the first conclusion to be drawn is that fractional order models are better in terms of the indices considered and the Akaike weight  $w$  calculated, although this may be considered unsurprising because they have more parameters, i.e., more flexibility for fitting. For this reason, the analysis of the results given in the previous section will be performed adopting different points of view: 1) the significance of the four variables of the model ( $x_1$ ,  $x_4$ ,  $x_5$  and  $x_6$ ) after fitting, 2) the values of the order for each variable in the fractional model, and 3) the values of coefficients  $C$ . For a better interpretation of the results given in Table 3, the orders  $\alpha$  of the model variables (except for  $x_1$ ) and the coefficients  $C$  of fractional model (3) are plotted on a EU map in Figs. 3 and 4, respectively. Although included in plots and tables, the results for HRV are omitted from the discussion for obvious reasons (only data for four years is available).

*Economic growth in the European Union modelled with fractional derivatives: first results*

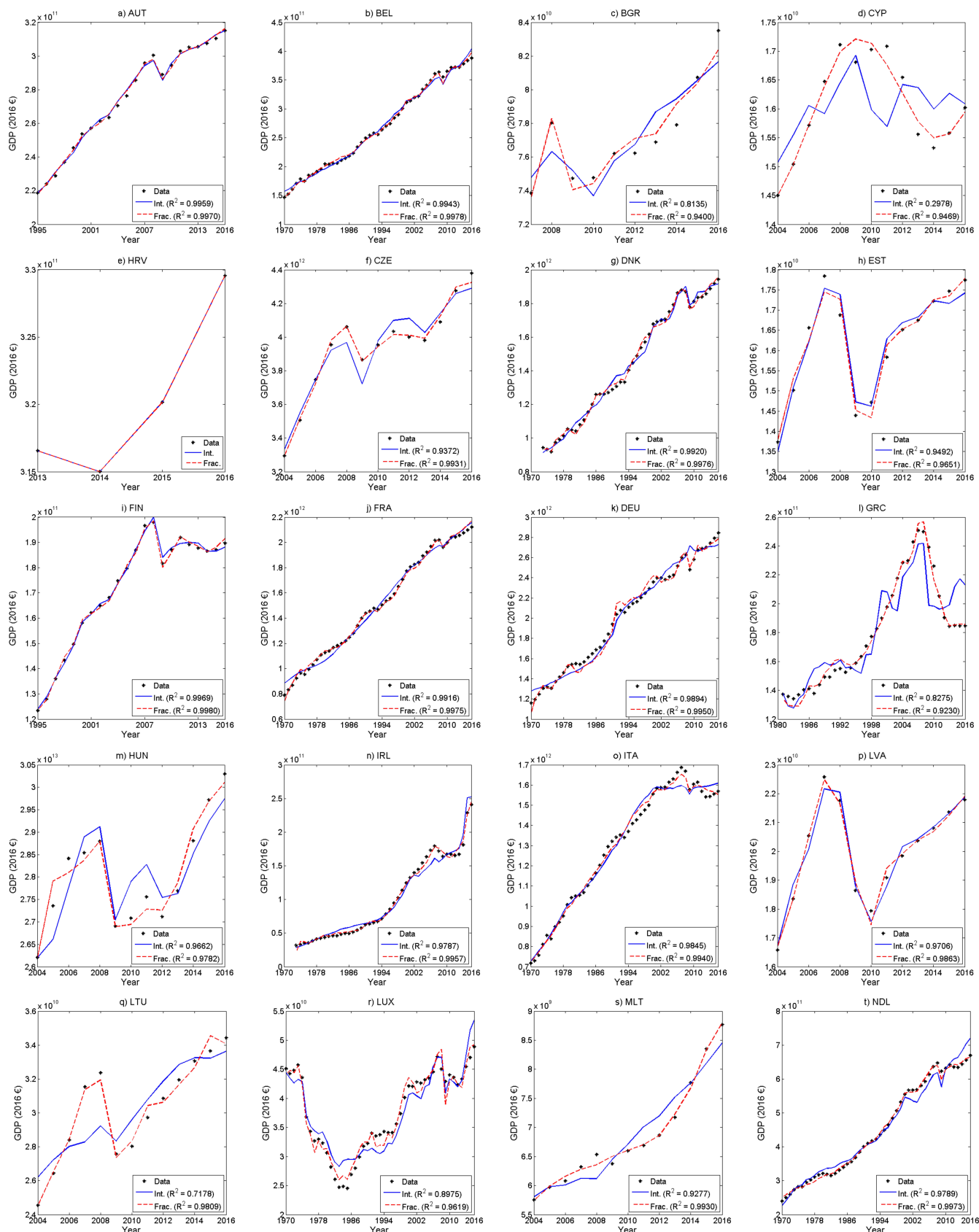


Fig. 1. Fitting results for integer model (2) and fractional model (3) for EU states: a) Austria, b) Belgium, c) Bulgaria, d) Cyprus, e) Croatia, f) Czech Republic, g) Denmark, h) Estonia, i) Finland, j) France, k) Germany, l) Greece, m) Hungary, n) Ireland, o) Italy, p) Latvia, q) Lithuania, r) Luxembourg, s) Malta, t) Netherlands (for illustration purposes, notice that the scale of  $y$ - and  $x$ -axis is not the same for all states)

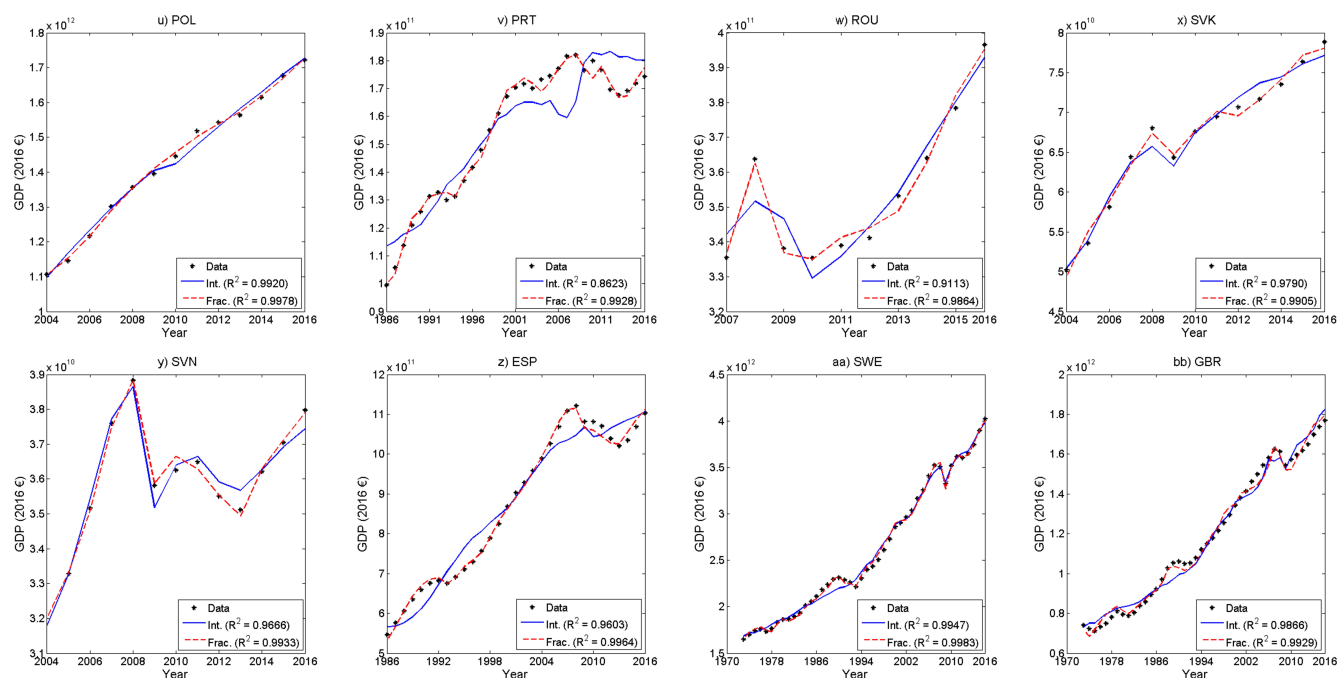


Fig. 2. Fitting results for integer model (2) and fractional model (3) for EU states (cont.): u) Poland, v) Portugal, w) Romania, x) Slovak Republic, y) Slovenia, z) Spain, aa) Sweden, bb) United Kingdom (for illustration purposes, notice that the scale of  $y$ - and  $x$ -axis is not the same for all states)

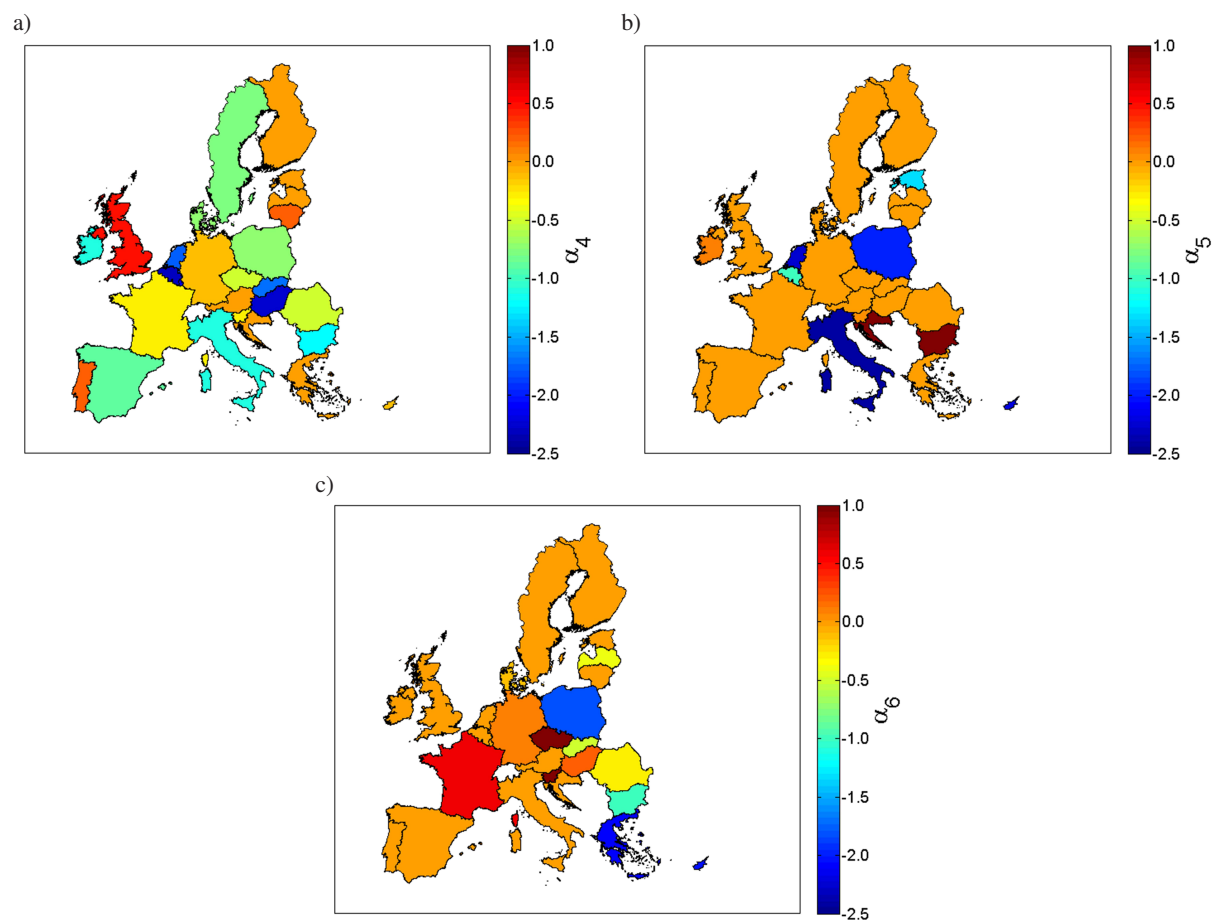


Fig. 3. Results for the orders of the variables of the fractional model (3) on an EU map: a) order  $\alpha_4$ , b) order  $\alpha_5$ , c) order  $\alpha_6$



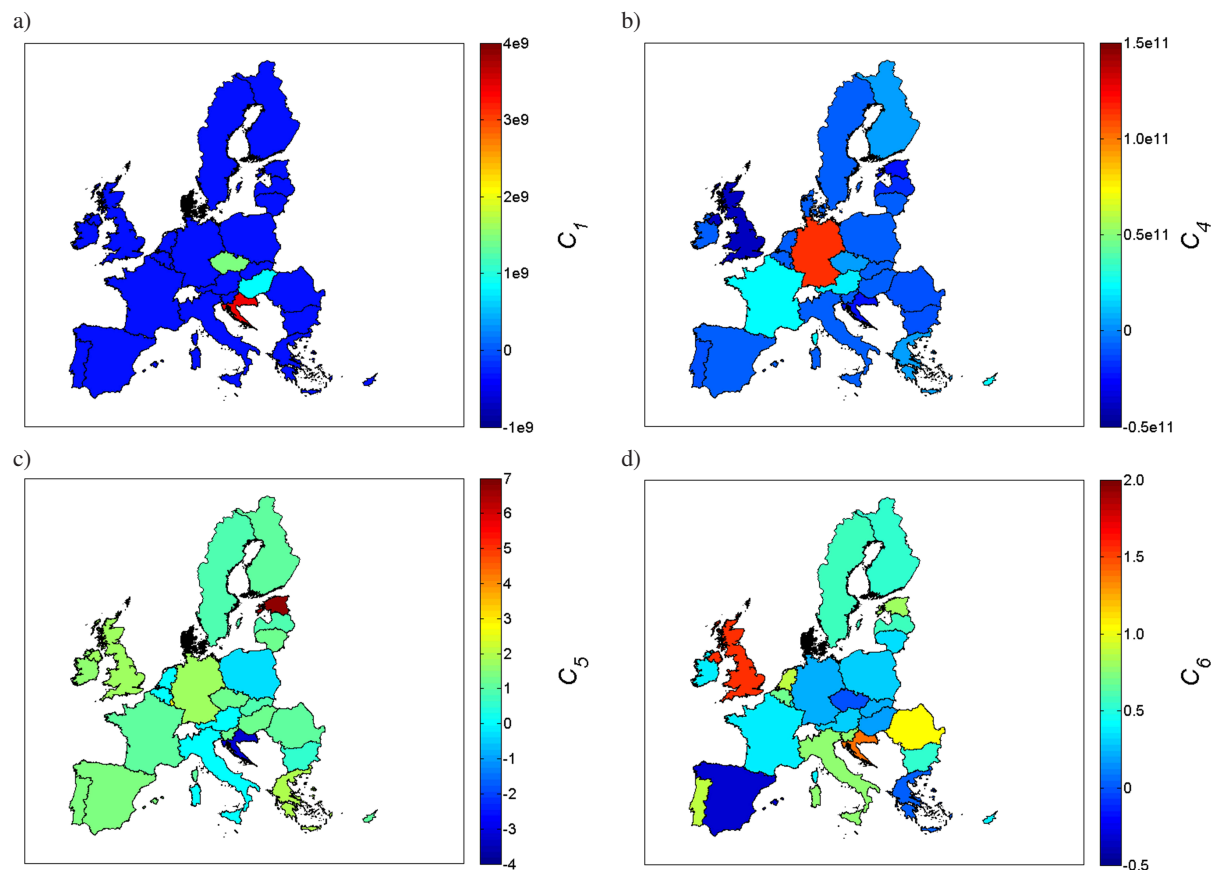


Fig. 4. Results for the coefficients of the fractional model (3) on an EU map: a) coefficient  $C_1$ , b) coefficient  $C_4$ , c) coefficient  $C_5$ , d) coefficient  $C_6$  (Denmark is colored black whenever her value is outside the range considered.)

As far as the significance of the variables of the model is concerned, it is observed that:

- For integer models of the form of (2), not all the variables are significant. This is true for all states, except for the group of five states formed by EST, FIN, HUN, LVA and SVN. Furthermore, for these five countries all the variables are also required for the fractional model.
- For fractional models of the form of (3), all the variables are significant. This is true for all states, except for the group of seven states which are (giving the variable, or variables, with no significance in brackets): AUT ( $x_5$ ), CYP ( $x_4$ ), CZE ( $x_6$ ), FRA ( $x_6$ ), GRC ( $x_1, x_4, x_6$ ), POL ( $x_4$ ), and ESP ( $x_6$ ).

Taking into account the orders of the fractional models (see Fig. 3):

- As might be expected, the order of variable  $x_1$  is always 0, with the exception of GRC, although even in this case the value of  $\alpha_1$  can be negligible. (It should be noticed that variable  $x_1$  is constant for all countries, except for DEU.)
- The models for AUT and FIN have all the orders equal to 0, i.e., the models for these states are of integer order instead. One reason for this result may be due to the fact that the integer order models can fit the data meaningfully well

(the value of the index  $R^2$  is higher than 0.99, the highest in the table). This circumstance makes the optimization process in MATLAB more difficult: more iterations, and consequently more time, are required to find the minimum of the MSE. However, the cause for the change of the order in  $x_5$  cannot be ascertained with ease.

- From Fig. 3b, it can be observed that the variable  $x_5$  has influence on GDP of three different forms depending on the value of its corresponding order as follows:

1. when  $\alpha_5 = -1$  (as for the integer model (2)), it is a measure of manufactured resources. This is the case for BEL. In the case of EST,  $\alpha_5 = -1.32$ . Furthermore, there is a group of countries with  $\alpha_5 = -2$ , or closer, which are ITA, MLT, NLD, and POL.
2. when  $\alpha_5 = 1$ , the effect of  $x_5$  is a measure of the impact of the variation of investment in the economy. This is the case of BGR.
3. when  $\alpha_5 = 0$ , which is the most common case obtained for the EU states (the remaining ones),  $x_5$  measures the impact of investment in the economy. In this group, it should be mentioned that  $\alpha_5$  is not exactly equal to 0 for GRC, IRL, PRT and SVN, but small and positive.

Table 2

Performance indices for integer model (2) and fractional model (3) for EU states (note for HRV: it is not possible to obtain  $t$ - and  $p$ -values since the matrix has more predictor variables than observations. These indices are marked as \*

Index	Variable	AUT		BEL		BGR		CYP		HRV		CZE		DNK	
		Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)
MSE		3.630 × 10 <sup>18</sup>	2.688 × 10 <sup>18</sup>	3.066 × 10 <sup>19</sup>	1.208 × 10 <sup>19</sup>	1.464 × 10 <sup>18</sup>	4.706 × 10 <sup>17</sup>	4.692 × 10 <sup>17</sup>	3.548 × 10 <sup>16</sup>	1.918 × 10 <sup>-6</sup>	0	4.852 × 10 <sup>21</sup>	5.338 × 10 <sup>20</sup>	8.989 × 10 <sup>30</sup>	2.643 × 10 <sup>30</sup>
R <sup>2</sup>		0.9959	0.9970	0.9943	0.9978	0.8135	0.9400	0.2978	0.9469	1	1	0.9372	0.9931	0.9920	0.9976
MAD		1.521 × 10 <sup>9</sup>	1.397 × 10 <sup>9</sup>	4.261 × 10 <sup>9</sup>	2.527 × 10 <sup>9</sup>	1.048 × 10 <sup>9</sup>	5.729 × 10 <sup>8</sup>	5.814 × 10 <sup>8</sup>	1.392 × 10 <sup>8</sup>	1.129 × 10 <sup>-3</sup>	0	5.755 × 10 <sup>10</sup>	1.833 × 10 <sup>10</sup>	2.498 × 10 <sup>10</sup>	1.398 × 10 <sup>10</sup>
<i>t</i> -values	<i>x</i> <sub>1</sub>	-1.589	-2.954	-3.704	34.464	-1.678	27.880	2.708	75.264	*	*	2.901	22.574	-1.600	57.835
	<i>x</i> <sub>4</sub>	4.444	5.868	11.169	-19.634	1.990	-4.863	0.904	-2.518	*	*	-2.518	21.339	12.923	18.992
	<i>x</i> <sub>5</sub>	-0.842	2.696	-0.273	16.505	-1.647	4.943	1.840	-6.705	*	*	-1.209	13.507	-3.186	18.350
	<i>x</i> <sub>6</sub>	6.154	2.914	5.385	11.059	1.347	5.052	-1.356	6.456	*	*	4.431	-1.508	7.690	-3.867
	<i>x</i> <sub>0</sub>														
<i>p</i> -values	<i>x</i> <sub>1</sub>	1.295 × 10 <sup>-1</sup>	8.488 × 10 <sup>-3</sup>	6.011 × 10 <sup>-4</sup>	5.899 × 10 <sup>-33</sup>	1.443 × 10 <sup>-1</sup>	1.409 × 10 <sup>-7</sup>	2.406 × 10 <sup>-2</sup>	6.518 × 10 <sup>-14</sup>	*	*	1.756 × 10 <sup>-2</sup>	3.113 × 10 <sup>-9</sup>	1.175 × 10 <sup>-1</sup>	3.556 × 10 <sup>-40</sup>
	<i>x</i> <sub>4</sub>	3.136 × 10 <sup>-4</sup>	1.479 × 10 <sup>-5</sup>	2.709 × 10 <sup>-14</sup>	4.336 × 10 <sup>-23</sup>	9.371 × 10 <sup>-2</sup>	2.813 × 10 <sup>-3</sup>	2.000 × 10 <sup>-1</sup>	3.897 × 10 <sup>-1</sup>	*	*	3.287 × 10 <sup>-2</sup>	5.123 × 10 <sup>-9</sup>	7.309 × 10 <sup>-16</sup>	1.273 × 10 <sup>-21</sup>
	<i>x</i> <sub>5</sub>	4.107 × 10 <sup>-1</sup>	1.479 × 10 <sup>-2</sup>	7.861 × 10 <sup>-1</sup>	3.209 × 10 <sup>-20</sup>	1.507 × 10 <sup>-1</sup>	2.596 × 10 <sup>-3</sup>	9.890 × 10 <sup>-2</sup>	8.800 × 10 <sup>-5</sup>	*	*	2.575 × 10 <sup>-1</sup>	2.793 × 10 <sup>-7</sup>	2.792 × 10 <sup>-3</sup>	4.384 × 10 <sup>-21</sup>
	<i>x</i> <sub>6</sub>	8.238 × 10 <sup>-6</sup>	9.257 × 10 <sup>-3</sup>	2.847 × 10 <sup>-6</sup>	3.723 × 10 <sup>-14</sup>	2.265 × 10 <sup>-1</sup>	2.327 × 10 <sup>-3</sup>	2.080 × 10 <sup>-1</sup>	1.173 × 10 <sup>-4</sup>	*	*	1.644 × 10 <sup>-3</sup>	1.659 × 10 <sup>-1</sup>	2.088 × 10 <sup>-9</sup>	3.961 × 10 <sup>-4</sup>
	<i>x</i> <sub>0</sub>														
AIC		950.5	943.9	2117.8	2074.0	434.3	422.9	542.0	508.4	*	*	662.1	633.44	2131.9	2078.1
w (%)		3.5	96.4	0	100	0.3	99.7	0	100	*	*	0	100	0	100
AIC without one variable	<i>x</i> <sub>1</sub>	950.4	949.6	2128.4	2229.3	432.1	465.6	545.4	587.8	*	*	666.4	681.81	2132.2	2270.9
	<i>x</i> <sub>4</sub>	963.8	964.4	2179.4	2180.0	433.3	432.9	540.1	504.8	*	*	664.7	680.4	2201.8	2177.0
	<i>x</i> <sub>5</sub>	948.4	948.4	2115.5	2165.3	432.0	433.1	541.8	527.3	*	*	659.8	668.8	2139.4	2174.3
	<i>x</i> <sub>6</sub>	972.4	949.4	2139.6	2134.9	430.9	4.33	540.0	526.5	*	*	672.8	632.0	2169.4	2089.6
w found from AIC	<i>x</i> <sub>1</sub>	30	25.3	0	0	23	0	3	0	*	*	3.3	0	97.4	0
	<i>x</i> <sub>4</sub>	0	0	0	0	12	38.2	39	100	*	*	7.4	0	0	0
	<i>x</i> <sub>5</sub>	70	46.9	100	0	24	33.6	17	0	*	*	89.2	0	2.6	0
	<i>x</i> <sub>6</sub>	0	27.8	0	100	41	28.2	41	0	*	*	0.1	100	0	100
Index	Variable	EST		FIN		FRA		DEU		GRC		HUN		IRL	
		Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)
MSE		8.534 × 10 <sup>16</sup>	5.858 × 10 <sup>16</sup>	1.512 × 10 <sup>18</sup>	9.720 × 10 <sup>17</sup>	1.409 × 10 <sup>21</sup>	4.168 × 10 <sup>20</sup>	4.677 × 10 <sup>21</sup>	2.555 × 10 <sup>21</sup>	2.247 × 10 <sup>20</sup>	1.734 × 10 <sup>19</sup>	2.468 × 10 <sup>23</sup>	5.365 × 10 <sup>22</sup>	7.494 × 10 <sup>19</sup>	1.515 × 10 <sup>19</sup>
R <sup>2</sup>		0.9492	0.9651	0.9969	0.9980	0.9916	0.9975	0.9894	0.9950	0.8275	0.9230	0.9662	0.9782	0.9957	0.9957
MAD		2.515 × 10 <sup>8</sup>	1.889 × 10 <sup>8</sup>	9.985 × 10 <sup>8</sup>	8.529 × 10 <sup>8</sup>	3.055 × 10 <sup>10</sup>	1.618 × 10 <sup>10</sup>	5.402 × 10 <sup>10</sup>	4.113 × 10 <sup>10</sup>	1.169 × 10 <sup>10</sup>	3.214 × 10 <sup>9</sup>	4.274 × 10 <sup>11</sup>	1.782 × 10 <sup>11</sup>	6.465 × 10 <sup>9</sup>	2.953 × 10 <sup>9</sup>
<i>t</i> -values	<i>x</i> <sub>1</sub>	8.535	9.862	-5.529	-5.400	1.781	24.167	6.015	9.922	-1.901	0.631	-5.984	17.757	-2.696	12.169
	<i>x</i> <sub>4</sub>	-8.102	-9.477	7.958	9.480	9.948	30.136	1.934	15.680	2.978	1.440	6.498	10.546	3.577	9.210
	<i>x</i> <sub>5</sub>	4.421	5.708	-6.332	8.508	1.405	9.233	10.644	13.606	-3.726	8.266	-10.236	9.270	-1.895	11.604
	<i>x</i> <sub>6</sub>	7.395	10.544	12.239	5.832	1.907	0.883	-6.546	4.532	5.607	1.696	9.380	6.718	7.585	13.179
	<i>x</i> <sub>0</sub>														
<i>p</i> -values	<i>x</i> <sub>1</sub>	1.314 × 10 <sup>-5</sup>	4.016 × 10 <sup>-6</sup>	3.000 × 10 <sup>-5</sup>	3.935 × 10 <sup>-5</sup>	8.200 × 10 <sup>-2</sup>	1.189 × 10 <sup>-26</sup>	3.474 × 10 <sup>-7</sup>	1.094 × 10 <sup>-12</sup>	6.633 × 10 <sup>-2</sup>	5.322 × 10 <sup>-1</sup>	2.065 × 10 <sup>-3</sup>	2.829 × 10 <sup>-8</sup>	1.022 × 10 <sup>-2</sup>	5.022 × 10 <sup>-15</sup>
	<i>x</i> <sub>4</sub>	2.000 × 10 <sup>-5</sup>	5.583 × 10 <sup>-6</sup>	2.641 × 10 <sup>-7</sup>	2.018 × 10 <sup>-8</sup>	8.102 × 10 <sup>-12</sup>	1.509 × 10 <sup>-30</sup>	5.967 × 10 <sup>-2</sup>	2.140 × 10 <sup>-19</sup>	5.496 × 10 <sup>-3</sup>	1.117 × 10 <sup>-1</sup>	2.295 × 10 <sup>-6</sup>	9.274 × 10 <sup>-4</sup>	1.963 × 10 <sup>-11</sup>	1.502 × 10 <sup>-14</sup>
	<i>x</i> <sub>5</sub>	1.668 × 10 <sup>-3</sup>	2.912 × 10 <sup>-4</sup>	5.750 × 10 <sup>-6</sup>	1.009 × 10 <sup>-7</sup>	1.673 × 10 <sup>-1</sup>	8.622 × 10 <sup>-12</sup>	1.256 × 10 <sup>-13</sup>	3.503 × 10 <sup>-17</sup>	7.517 × 10 <sup>-4</sup>	1.921 × 10 <sup>-9</sup>	2.946 × 10 <sup>-6</sup>	6.700 × 10 <sup>-6</sup>	6.528 × 10 <sup>-2</sup>	2.236 × 10 <sup>-14</sup>
	<i>x</i> <sub>6</sub>	4.123 × 10 <sup>-5</sup>	2.298 × 10 <sup>-6</sup>	3.670 × 10 <sup>-10</sup>	1.593 × 10 <sup>-5</sup>	6.321 × 10 <sup>-2</sup>	3.820 × 10 <sup>-1</sup>	5.862 × 10 <sup>-8</sup>	4.613 × 10 <sup>-5</sup>	3.396 × 10 <sup>-6</sup>	9.949 × 10 <sup>-2</sup>	6.081 × 10 <sup>-6</sup>	8.676 × 10 <sup>-5</sup>	2.906 × 10 <sup>-9</sup>	3.862 × 10 <sup>-17</sup>
	<i>x</i> <sub>0</sub>														
AIC		519.8	514.9	931.3	921.5	2297.7	2240.5	2354.1	2325.7	1696.3	1604.1	713.2	693.4	2022.6	1952.3
w (%)		8	92	0.8	99.2	0	100	0	100	0	100	0	100	0	100
AIC without one variable	<i>x</i> <sub>1</sub>	544.2	542.7	950.1	939.7	2298.7	2364.0	2362.6	2346.5	1697.6	1665.1	707.1	7353.8	2027.5	2018.0
	<i>x</i> <sub>4</sub>	543.0	541.7	961.4	957.9	2351.5	2383.6	2354.8	2376.7	1702.6	1725.8	708.9	7227.4	2032.4	2000.0
	<i>x</i> <sub>5</sub>	530.5	530.5	954.0	954.0	2297.4	2289.5	2360.6	2360.6	1706.7	1720.5	719.2	7196.7	2024.0	2014.7
	<i>x</i> <sub>6</sub>	540.9	544.3	977.4	941.9	2299.1	2238.9	2384.8	2330.2	1718.4	1604.4	724.7	7123.7	2059.4	2032.6
w found from AIC	<i>x</i> <sub>1</sub>	0.1	0.2	87.5	74.5	27.4	0	2	0	91.4	0	70.3	0	14.2	0
	<i>x</i> <sub>4</sub>	0.2	0.4	0.3	0	0	0	98	0	7.7	0	29.5	0.5	1.3	99.9
	<i>x</i> <sub>5</sub>	99.2	99.3	12.2	0.1	50.9	0	0	0	0.9	0	0.2	2.5	84.5	0.1
	<i>x</i> <sub>6</sub>	0.5	0.1	0	25.4	21.7	100	0	100	0	100	0	97	0	0
Index	Variable	ITA		LVA		LTU		LUX		MLT		NLD		POL	
		Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)
MSE		1.357 × 10 <sup>21</sup>	5.234 × 10 <sup>20</sup>	8.403 × 10 <sup>16</sup>	3.923 × 10 <sup>16</sup>	2.371 × 10 <sup>18</sup>	1.601 × 10 <sup>17</sup>	5.063 × 10 <sup>18</sup>	1.880 × 10 <sup>18</sup>	5.634 × 10 <sup>16</sup>	5.438 × 10 <sup>15</sup>	4.127 × 10 <sup>20</sup>	5.308 × 10 <sup>19</sup>	2.901 × 10 <sup>30</sup>	8.156 × 10 <sup>19</sup>
R <sup>2</sup>		0.9845	0.9940	0.9706	0.9863	0.7178	0.9809	0.8975	0.9619	0.9277	0.9930	0.9789	0.9973	0.9920	0.9978
MAD		3.010 × 10 <sup>10</sup>	1.998 × 10 <sup>10</sup>	2.432 × 10 <sup>8</sup>	1.468 × 10 <sup>8</sup>	1.225 × 10 <sup>9</sup>	3.176 × 10 <sup>8</sup>	1.896 × 10 <sup>9</sup>	1.093 × 10 <sup>9</sup>	1.918 × 10 <sup>8</sup>	5.219 × 10 <sup>7</sup>	1.548 × 10 <sup>10</sup>	5.901 × 10 <sup>9</sup>	1.394 × 10 <sup>10</sup>	7.590 × 10 <sup>9</sup>
<i>t</i> -values	<i>x</i> <sub>1</sub>	-10.502	50.626	11.998	5.289	0.131	23.507	0.890	13.121	-0.282	7.154	-2.871	36.489	-3.041	120.108
	<i>x</i> <sub>4</sub>	15.350	18.160	-11.692	-4.820	$6.493 \times 10^{-2}$	-3.214	-8.233 × 10 <sup>-2</sup>	-7.463	0.610	-6.226	4.267	14.807	3.927	0.838
	<i>x</i> <sub>5</sub>	-11.023	-25.825	7.020	12.932	-1.247 × 10 <sup>-2</sup>	10.208	-1.384	8.194	0.507	-5.153	-1.761	-25.323	3.697	-3.630
	<i>x</i> <sub>6</sub>	1.875	4.882	8.067	8.055	1.058	17.500	11.677	7.454	-0.721	5.580	3.984	11.808	-0.778	3.637
	<i>x</i> <sub>0</sub>														
<i>p</i> -values	<i>x</i> <sub>1</sub>	$1.912 \times 10^{-13}$	$5.784 \times 10^{-40}$	$7.712 \times 10^{-7}$	$5.010 \times 10^{-4}$	$8.989 \times 10^{-1}$	$2.174 \times 10^{-9}$	$3784 \times 10^{-1}$	$1.242 \times 10^{-16}$	$7.841 \times 10^{-1}$	$5.342 \times 10^{-5}$	$6.327 \times 10^{-3}$	$5.486 \times 10$		

Table 3

Fitting results for integer model (2) and fractional model (3) for EU states: coefficients and orders of the fractional operator (notice that the orders  $\alpha_k$  for integer model are  $\alpha_1 = 0$ ,  $\alpha_4 = 0$ ,  $\alpha_5 = -1$  and  $\alpha_6 = 0$ )

AUT					BEL				
Int. (2)	$C_k$	$-1.122 \times 10^6$	$3.082 \times 10^{10}$	$-4.117 \times 10^{-3}$	$5.350 \times 10^{-1}$	Int. (2)	$C_k$	$-2.672 \times 10^6$	$2.665 \times 10^{10}$
Frac. (3)	$C_k$	$-2.107 \times 10^6$	$3.667 \times 10^{10}$	$7.373 \times 10^{-2}$	$3.065 \times 10^{-1}$	Frac. (3)	$C_k$	$3.706 \times 10^6$	$-9.664 \times 10^6$
	$\alpha_k$	0	0	0	0		$\alpha_k$	0	-2.31
BGR					CYP				
Int. (2)	$C_k$	$-3.112 \times 10^6$	$3.754 \times 10^{10}$	$-8.978 \times 10^{-2}$	$2.743 \times 10^{-1}$	Int. (2)	$C_k$	$4.559 \times 10^6$	$-1.502 \times 10^9$
Frac. (3)	$C_k$	$5.674 \times 10^5$	$-1.202 \times 10^9$	$4.636 \times 10^{-1}$	$5.279 \times 10^{-1}$	Frac. (3)	$C_k$	$8.945 \times 10^5$	$4.230 \times 10^{10}$
Orders	$\alpha_k$	0	-1.25	1	-1	Orders	$\alpha_k$	0	-0.10
HRV					CZE				
Int. (2)	$C_k$	$4.028 \times 10^8$	$-1.845 \times 10^{12}$	2.542	$6.607 \times 10^{-1}$	Int. (2)	$C_k$	$2.149 \times 10^9$	$-1.142 \times 10^{12}$
Frac. (3)	$C_k$	$6.164 \times 10^8$	$-1.941 \times 10^{10}$	-3.296	1.416	Frac. (3)	$C_k$	$2.874 \times 10^8$	$1.580 \times 10^{10}$
	$\alpha_k$	0	$2.50 \times 10^{-4}$	1	0		$\alpha_k$	0	-0.52
DNK					EST				
Int. (2)	$C_k$	$-2.833 \times 10^6$	$1.086 \times 10^{11}$	$-3.894 \times 10^{-2}$	1.104	Int. (2)	$C_k$	$4.701 \times 10^6$	$-1.595 \times 10^{10}$
Frac. (3)	$C_k$	$3.938 \times 10^9$	$1.008 \times 10^9$	$9.430 \times 10^{10}$	-5.001	Frac. (3)	$C_k$	$5.602 \times 10^6$	$-1.911 \times 10^{10}$
	$\alpha_k$	0	-0.52	0	1		$\alpha_k$	0	-1.32
FIN					FRA				
Int. (2)	$C_k$	$-5.971 \times 10^5$	$2.963 \times 10^{10}$	$-2.556 \times 10^{-2}$	$9.395 \times 10^{-1}$	Int. (2)	$C_k$	$2.424 \times 10^5$	$13.160 \times 10^{10}$
Frac. (3)	$C_k$	$-3.153 \times 10^5$	$1.886 \times 10^{10}$	1.020	$5.326 \times 10^{-1}$	Frac. (3)	$C_k$	$8.945 \times 10^5$	$4.230 \times 10^{10}$
	$\alpha_k$	0	0	0	0		$\alpha_k$	0	-0.3
DEU					GRC				
Int. (2)	$C_k$	$9.862 \times 10^5$	$1.520 \times 10^{11}$	$5.359 \times 10^{-2}$	$-4.703 \times 10^{-1}$	Int. (2)	$C_k$	$-1.448 \times 10^6$	$3.571 \times 10^{10}$
Frac. (3)	$C_k$	$1.763 \times 10^6$	$1.568 \times 10^{11}$	1.847	$2.014 \times 10^{-1}$	Frac. (3)	$C_k$	$-4.889 \times 10^5$	$1.742 \times 10^{10}$
	$\alpha_k$	0	-0.1	0	0.1		$\alpha_k$	$1.25 \times 10^{-5}$	0
HUN					IRL				
Int. (2)	$C_k$	$2.070 \times 10^8$	0	$-9.426 \times 10^{-2}$	$6.105 \times 10^{-1}$	Int. (2)	$C_k$	$-1.676 \times 10^6$	$1.648 \times 10^{10}$
Frac. (3)	$C_k$	$1.921 \times 10^8$	$2.362 \times 10^9$	1.262	$1.732 \times 10^{-1}$	Frac. (3)	$C_k$	$2.858 \times 10^5$	$9.213 \times 10^7$
	$\alpha_k$	0	-2.29	0	0.25		$\alpha_k$	0	-1.1
ITA					LVA				
Int. (2)	$C_k$	$-5.227 \times 10^6$	$3.824 \times 10^{11}$	$-1.116 \times 10^{-1}$	$5.085 \times 10^{-1}$	Int. (2)	$C_k$	$4.971 \times 10^6$	$-2.871 \times 10^{10}$
Frac. (3)	$C_k$	$2.257 \times 10^6$	$3.260 \times 10^9$	$-1.800 \times 10^{-3}$	$7.875 \times 10^{-1}$	Frac. (3)	$C_k$	$1.907 \times 10^6$	$-1.015 \times 10^{10}$
	$\alpha_k$	0	-1.12	-2.42	0		$\alpha_k$	0	0
LTU					LUX				
Int. (2)	$C_k$	$2.306 \times 10^5$	$6.599 \times 10^8$	$-2.440 \times 10^{-3}$	$4.206 \times 10^{-1}$	Int. (2)	$C_k$	$1.045 \times 10^7$	$-3.287 \times 10^8$
Frac. (3)	$C_k$	$2.911 \times 10^5$	$-2.339 \times 10^8$	1.230	$3.510 \times 10^{-1}$	Frac. (3)	$C_k$	$4.585 \times 10^6$	$-1.378 \times 10^6$
	$\alpha_k$	0	0.22	0	$1.31 \times 10^{-3}$		$\alpha_k$	0	-2.20
MLT					NLD				
Int. (2)	$C_k$	$-1.987 \times 10^7$	$1.251 \times 10^8$	$1.424 \times 10^{-1}$	$-2.038 \times 10^{-1}$	Int. (2)	$C_k$	$-1.502 \times 10^7$	$7.902 \times 10^{-10}$
Frac. (3)	$C_k$	$1.395 \times 10^8$	$-3.685 \times 10^9$	$-5.846 \times 10^{-1}$	$1.557 \times 10^{-1}$	Frac. (3)	$C_k$	$5.840 \times 10^6$	$1.191 \times 10^8$
	$\alpha_k$	0	0	-2.00	-1.73		$\alpha_k$	0	-1.80
POL					PRT				
Int. (2)	$C_k$	$-1.220 \times 10^7$	$4.350 \times 10^{11}$	$1.239 \times 10^{-1}$	$-2.221 \times 10^{-1}$	Int. (2)	$C_k$	$1.884 \times 10^5$	$1.879 \times 10^{10}$
Frac. (3)	$C_k$	$3.605 \times 10^6$	$1.729 \times 10^9$	$-3.696 \times 10^{-1}$	$2.853 \times 10^{-1}$	Frac. (3)	$C_k$	$9.288 \times 10^5$	$-3.980 \times 10^9$
	$\alpha_k$	0	-0.70	-2.00	-1.82		$\alpha_k$	0	0.22
ROU					SVK				
Int. (2)	$C_k$	$1.322 \times 10^7$	$-2.535 \times 10^{11}$	$2.389 \times 10^{-1}$	$-6.986 \times 10^{-2}$	Int. (2)	$C_k$	$-5.295 \times 10^6$	$2.413 \times 10^{10}$
Frac. (3)	$C_k$	$9.832 \times 10^5$	$-4.304 \times 10^9$	$9.883 \times 10^{-1}$	1.056	Frac. (3)	$C_k$	$7.653 \times 10^5$	$-3.000 \times 10^7$
	$\alpha_k$	0	-0.52	0	-0.30		$\alpha_k$	0	-1.72
SVN					ESP				
Int. (2)	$C_k$	$1.681 \times 10^7$	$-2.720 \times 10^{10}$	$1.210 \times 10^{-1}$	$7.472 \times 10^{-1}$	Int. (2)	$C_k$	$-7.682 \times 10^6$	$1.373 \times 10^{11}$
Frac. (3)	$C_k$	$9.491 \times 10^5$	$6.290 \times 10^8$	1.159	$7.646 \times 10^{-1}$	Frac. (3)	$C_k$	$8.059 \times 10^5$	$2.515 \times 10^9$
	$\alpha_k$	0	-0.28	0.04	1		$\alpha_k$	0	-0.88
SWE					GBR				
Int. (2)	$C_k$	$2.640 \times 10^6$	$3.673 \times 10^{10}$	$3.397 \times 10^{-2}$	$7.826 \times 10^{-1}$	Int. (2)	$C_k$	$4.238 \times 10^6$	$-5.823 \times 10^{10}$
Frac. (3)	$C_k$	$2.654 \times 10^6$	$3.401 \times 10^9$	$9.186 \times 10^{-1}$	$5.846 \times 10^{-1}$	Frac. (3)	$C_k$	$1.910 \times 10^6$	$-3.845 \times 10^{10}$
	$\alpha_k$	0	-0.8	0	0		$\alpha_k$	0	0.5

Table 4  
Economic data for all states of the EU in the period 1970–2016

[illegible]



Table 5  
Economic data for all states of the EU in the period 1970–2016 (cont)

[illegible]



With respect the coefficients  $C$  (see Fig. 4), the following can be concluded:

- There is some uniformity in coefficient  $C_1$  (Fig. 4a), except for DNK (it is represented in black because is out of range), CZE, and HUN. This may mean that the land area has a similar influence on GDP for each country.
- For  $C_4$  (Fig. 4b), there is less uniformity than in the previous case, but still some, except mainly in the case of DEU, and less of FRA and AUT.
- Although with some exceptions for center and Eastern states, the values of coefficient  $C_5$  seem to be regular (Fig. 4c).
- There is no apparent pattern for  $C_6$  (Fig. 4d) for the whole EU, or even for regions.

To sum up, we consider that an in-depth study is needed in order to obtain more conclusive results concerning economic growth modelling of all states of the EU, since either 1970 or the year of accession to the EU to 2016. The consideration of other variables, such as all those considered in [28, 29], may be required for a better description of GDP.

## A. Appendix: Data

The economic data used in this work can be found in Tables 4 and 5. Sources: data for  $x_1$ ,  $x_5$  and  $x_6$  can be found in World Bank database [31], whereas data for variable  $x_4$  was obtained for most of EU countries from Lee-Lee dataset [32], except for Croatia, Estonia, Latvia, Lithuania, Slovenia and Slovak Republic; for those six countries, Barro-Lee database [33] was used instead. Since data for  $x_4$  are included in both databases, it should be mentioned that: 1) it was presented every 5 years, so a third-order spline interpolation was used to obtain the data for every year; and 2) it was available for the period 1970–2010, so was extended to 2015 using the Wittgenstein projection from [34] (the last available value of [32] or [33] was extrapolated using the increase rate of the spline interpolation of [34]).

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